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Spectrum in the presence of brane-localized mass on torus extra dimensions

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ABSTRACT: The lightest mass eigenvalue of a six-dimensional theory compactified on a torus is numerically evaluated in the presence of the brane-localized mass term. The dependence on the cutoff scale Λ is non-negligible even when Λ is two orders of magnitude above the compactification scale, which indicates that the mass eigenvalue is sensitive to the size of the brane, in contrast to five-dimensional theories. We obtain an approximate expression of the lightest mass in the thin brane limit, which well fits the numerical calculations, and clarifies its dependence on the torus moduli parameter τ . We find that the lightest mass is typically much lighter than the compactification scale by an order of magnitude even in the limit of a large brane mass.

KEYWORDS: Phenomenology of Field Theories in Higher Dimensions, Strings and branes phenomenology

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1 Introduction

Many extra-dimensional models have four-dimensional (4D) brane-like defects on the compact space, such as orbifold fixed points or solitonic objects [1]–[4]. We can freely introduce 4D terms localized at the branes¹ [5–7]. Such brane-localized terms are induced by quantum effect even if they are absent at tree level [8, 9]. They change the Kaluza-Klein (KK) spectrum and deform the profiles of the mode functions [10–12]. In particular, the brane-localized mass terms are often introduced in order to remove unwanted modes from the 4D effective theory [13–15]. In five-dimensional (5D) theories, the effects of such brane masses can be translated into the change of the boundary conditions for the bulk fields. This is because the branes in 5D can be regarded as the boundaries of the extra dimension. In this case, large brane masses can make zero-modes of the bulk fields heavy enough up to half of the compactification scale.

In contrast, the branes are no longer the boundaries of the extra compact space in higher-dimensional theories. Since effects of the brane terms spread over higher-dimensional space and are diluted, they are expected to be smaller than those in the 5D case. Therefore, it is important to check whether the brane mass can make unwanted modes heavy enough or

¹Here we do not consider branes spread over other dimensions. The word “brane” is understood as the “3-brane” in this paper.

not. In this paper, we evaluate the lightest mass eigenvalue of a six-dimensional (6D) theory in the presence of the brane-localized mass term. The authors of ref. [11] discussed a closely related issue in the case of the T^2/Z_2 compactification whose torus moduli parameter is $\tau = i$, and obtained the result that the inverse of the lightest mass eigenvalue has a logarithmic dependence on the cutoff scale. Here we generalize their setup and consider a generic torus whose moduli parameter is arbitrary. Then we can explicitly see the relation to the well-known results in the 5D theories by squashing or stretching the torus. Besides, we are interested in a different parameter region from that discussed in ref. [11]. We mainly focus on the limit of a large brane mass, in which the dependence of the mass eigenvalues on the brane mass is negligible, and evaluate the ratio of the lightest mass to the compactification scale by numerical calculations.

The paper is organized as follows. After explaining the setup in the next section, we will see the dependences of the lightest mass eigenvalue on the cutoff scale of the theory and on the brane mass in section 3. In section 4, we find an approximate expression of the lightest mass as a function of the torus moduli parameter τ , and estimate its ratio to the compactification scale. Section 5 is devoted to the summary. We provide a brief review of the case of a 5D theory in appendix A, and discuss theories with fermion or vector field in appendix B.

2 Setup

We consider a 6D theory of a complex scalar field ϕ as a simple example.² The Lagrangian is given by

$$\mathcal{L} = -\partial^M \phi^* \partial_M \phi - c^2 |\phi|^2 \delta(x^4) \delta(x^5) + \dots, \quad (2.1)$$

where $M = 0, 1, 2, \dots, 5$, and the ellipsis denotes interaction terms, which are irrelevant to the following discussion. The brane mass parameter c is a real dimensionless constant. The extra dimensions are compactified on a torus T^2 .³ The background metric is assumed to be flat, for simplicity. For the coordinates of the extra dimensions, it is convenient to use a complex (dimensionless) coordinate $z \equiv \frac{1}{2\pi R}(x^4 + ix^5)$, where $R > 0$ is one of the radii of T^2 . The torus is defined by identifying points in the extra dimensions as

$$z \sim z + n_1 + n_2 \tau, \quad (n_1, n_2 \in \mathbb{Z}) \quad (2.2)$$

where τ is a complex constant that satisfies $\text{Im } \tau > 0$.

The Lagrangian (2.1) is then rewritten as

$$\mathcal{L} = -\partial^\mu \phi^* \partial_\mu \phi - \frac{1}{2(\pi R)^2} \left\{ |\partial_z \phi|^2 + |\partial_{\bar{z}} \phi|^2 + c^2 |\phi|^2 \delta^{(2)}(z) \right\} + \dots, \quad (2.3)$$

where $\mu = 0, 1, 2, 3$, and we have used that

$$\delta(x^4) \delta(x^5) = \frac{1}{2(\pi R)^2} \delta^{(2)}(z). \quad (2.4)$$

²Cases of fermion and vector fields are briefly discussed in appendix B.

³The spectrum in the case of T^2/Z_N compactification ($N = 2, 3, 4, 6$) can easily be obtained by thinning out the spectrum on T^2 .

a	0	1	2	3	4	5	6	7
(n, l)	(0,0)	(1, 0)	(0, 1)	(0, -1)	(-1, 0)	(1, 1)	(1, -1)	(-1, 1)
$\tilde{\lambda}_a$	0	3.14	3.14	3.14	3.14	4.44	4.44	4.44
a	8	9	10	11	12	13	14	\dots
(n, l)	(-1, -1)	(2,0)	(0,2)	(0, -2)	(-2, 0)	(2,1)	(2, -1)	\dots
$\tilde{\lambda}_a$	4.44	6.28	6.28	6.28	6.28	7.02	7.02	\dots

Table 1. Relabeling the KK modes in the case of $\tau = i$.

We can expand ϕ as

$$\phi(x^\mu, z) = \sum_{n,l=-\infty}^{\infty} f_{n,l}(z) \phi_{n,l}(x^\mu), \quad (2.5)$$

where

$$f_{n,l}(z) = \frac{1}{2\pi R \sqrt{\text{Im } \tau}} \exp \left\{ \frac{2\pi i}{\text{Im } \tau} \text{Im} \{ (n + l\bar{\tau})z \} \right\} \quad (2.6)$$

are normalized as

$$\begin{aligned} \int_{T^2} dx^4 dx^5 \left| f_{n,l} \left(\frac{x^4 + ix^5}{2\pi R} \right) \right|^2 &= 2(\pi R)^2 \int d^2 z |f_{n,l}(z)|^2 \\ &= (2\pi R)^2 \text{Im } \tau \int_0^1 dw_1 \int_0^1 dw_2 |f_{n,l}(w_1 + \tau w_2)|^2 = 1, \end{aligned} \quad (2.7)$$

and satisfy

$$\partial_z \partial_{\bar{z}} f_{n,l} = -\tilde{\lambda}_{n,l}^2 f_{n,l}, \quad \tilde{\lambda}_{n,l} = \frac{\pi |n + l\tau|}{\text{Im } \tau}. \quad (2.8)$$

This corresponds to the KK expansion in the absence of the brane-localized mass term. The KK masses are given by $\tilde{m}_{n,l} \equiv \tilde{\lambda}_{n,l}/(\pi R)$.

Since the 6D theory is non-renormalizable, it should be regarded as an effective theory valid only below the cutoff scale Λ . Here we relabel the KK modes by using the KK label $a = 0, 1, 2, \dots$ defined in such a way that

$$0 = \tilde{m}_0 < \tilde{m}_1 \leq \tilde{m}_2 \leq \dots \leq \tilde{m}_{N_\Lambda} < \Lambda \leq \tilde{m}_{N_\Lambda+1} \leq \dots \quad (2.9)$$

The correspondence of the labels (n, l) and a depends on the value of τ , as shown in tables 1 and 2. The number of the KK excited modes below Λ , i.e., N_Λ , grows as

$$N_\Lambda \propto \Lambda^2, \quad (2.10)$$

except for regions: $\arg \tau \simeq 0, \pi$, $|\tau| \ll 1$ and $|\tau| \gg 1$, in which the spacetime approaches 5D and thus $N_\Lambda \propto \Lambda$.

Then, (2.5) is rewritten as

$$\phi(x^\mu, z) = \sum_{a=0}^{\infty} f_a(z) \phi_a(x^\mu). \quad (2.11)$$

a	0	1	2	3	4	5	6	7
(n, l)	(0,0)	(2, -1)	(-2, 1)	(1, 0)	(-1, 0)	(1, -1)	(-1, 1)	(3, -1)
$\tilde{\lambda}_a$	0	3.20	3.20	4.10	4.10	4.69	4.69	5.68
a	8	9	10	11	12	13	14	\dots
(n, l)	(-3, 1)	(4, -2)	(-4, 2)	(3, -2)	(-3, 2)	(2, 0)	(0, 1)	\dots
$\tilde{\lambda}_a$	5.68	6.41	6.41	6.90	6.90	8.21	8.21	\dots

Table 2. Relabeling the KK modes in the case of $\tau = 2 \exp(\pi i/8)$.

Plugging (2.11) into (2.3) and performing the d^2z -integral, we obtain the 4D Lagrangian:

$$\mathcal{L}^{(4D)} = - \sum_a \partial^\mu \phi_a^* \partial_\mu \phi_a - \sum_{a,b} M_{ab}^2 \phi_a^* \phi_b + \dots, \quad (2.12)$$

where

$$\begin{aligned} M_{ab}^2 &\equiv \frac{\tilde{\lambda}_a^2}{\pi^2 R^2} \delta_{ab} + c^2 f_a^*(0) f_b(0) \\ &= \tilde{m}_a^2 \delta_{ab} + \frac{c^2}{4\pi^2 R^2 \text{Im } \tau} \end{aligned} \quad (2.13)$$

is the mass matrix of our theory.

3 Cutoff dependence

Since the theory is valid below Λ , we only consider the KK modes ϕ_a ($a = 0, 1, \dots, N_\Lambda$). Then, the mass squared eigenvalues, which are denoted as $\{m_0^2, m_1^2, \dots, m_{N_\Lambda}^2\}$, are obtained as eigenvalues of the finite matrix M_{ab}^2 ($a, b = 0, 1, \dots, N_\Lambda$).

Since $\tilde{m}_{n,l}^2 = \tilde{m}_{-n,-l}^2$, all the nonzero modes have degenerate modes when the brane mass is absent. Especially, $\tilde{m}_1^2 = \tilde{m}_2^2$. This means that M_{ab}^2 has the eigenvalue \tilde{m}_1^2 with the eigenvector $(0, 1, -1, 0, 0, \dots, 0)$. In fact, this is the second smallest eigenvalue of \tilde{M}_{ab}^2 . Namely, the mass of the first KK excited mode m_1 is independent of c and Λ :

$$m_1 = \tilde{m}_1 = \frac{1}{R \text{Im } \tau} \cdot \min_{(n,l) \neq (0,0)} |n + l\tau|. \quad (3.1)$$

Thus we take m_1 as the compactification scale throughout the paper.

Plots in figure 1 show the Λ -dependence of the lightest eigenvalue m_0 in the cases of $\tau = \exp(\frac{\pi i}{120})$, $\exp(\frac{2\pi i}{3})$, and $50 \exp(\frac{2\pi i}{3})$ and $c = 10.0$, in the unit of m_1 . The right end of the horizontal axis in each plot corresponds to the value of Λ such that $N_\Lambda \simeq 4000$. For a given value of c , the ratio m_0/m_1 can be approximated by

$$\frac{m_0}{m_1} \simeq \left(\alpha_1 + \alpha_2 \ln \frac{\Lambda}{m_1} + \alpha_3 \frac{\Lambda}{m_1} \right)^{-1} + \alpha_4, \quad (3.2)$$

where α_i ($i = 1, 2, 3, 4$) are real constants. The solid lines in figure 1 represent the fitting functions of the form (3.2). The constant α_4 is the asymptotic value of m_0/m_1 in the limit of $\Lambda \rightarrow \infty$:

$$\lim_{\Lambda \rightarrow \infty} \frac{m_0(\Lambda)}{m_1} = \alpha_4. \quad (3.3)$$

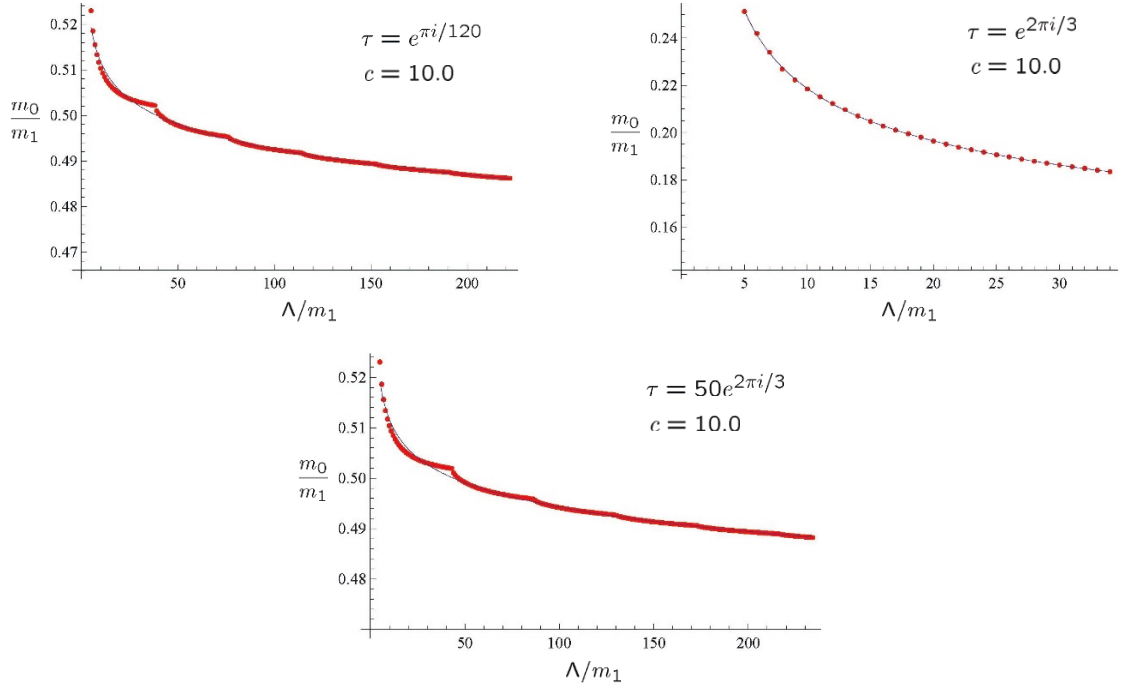


Figure 1. The lightest mass eigenvalue m_0 as a function of Λ in the case of $\tau = \exp(\pi i/120)$, $\exp(2\pi i/3)$ and $50\exp(2\pi i/3)$ and $c = 10.0$. The solid lines represent the function (3.2) with the parameters $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (11.9, 4.01, 0.0728, 0.466)$, $(1.82, 3.69, 0.270, 0.142)$ and $(12.4, 4.72, 0.0701, 0.470)$, respectively.

The horizontal axes in figure 1 denote the asymptotic lines that the curves approach. Typically, m_0 approaches to the limit value much more slowly compared with the 5D case (see figure 6 in appendix A.2). Thus the cutoff dependence of the spectrum cannot be neglected even when $\Lambda/m_1 = \mathcal{O}(100)$. This cutoff dependence becomes smaller when $\arg \tau \simeq 0, \pi$ or $|\tau| \ll 1$ or $|\tau| \gg 1$. This is because the torus is squashed or stretched in such cases, and the spacetime approaches to 5D. In fact, as we can see from figure 1,

$$\frac{m_0(15m_1)}{m_1} \simeq \lim_{\Lambda \rightarrow \infty} \frac{m_0(\Lambda)}{m_1} \times \begin{cases} 1.40 & (\tau = e^{2\pi i/3}) \\ 1.07 & (\tau = e^{\pi i/120}, 50e^{2\pi i/3}) \end{cases}. \quad (3.4)$$

Note that the curve for $\Lambda < 40m_1$ in the top-left plot or in the bottom plot are almost the same as that of the 5D case (shown in figure 6). The cusp at $\Lambda = 40m_1$ indicates that the field begins to feel the width of the squashed torus or the smaller cycle of the long thin torus.

In the following, we focus on the limit value (3.3). Figure 2 shows its dependence on the brane mass c . The unit here is taken as $1/(\pi R)$. For small values of c , the lightest mass eigenvalue m_0 is approximated as

$$m_0 \simeq \sqrt{M_{00}^2} = \frac{c}{2\pi R \sqrt{\text{Im} \tau}}, \quad (3.5)$$

which is plotted as the solid line in figure 2. This is because the brane mass can be treated as a perturbation in this region, and the mixing among the KK modes induced by

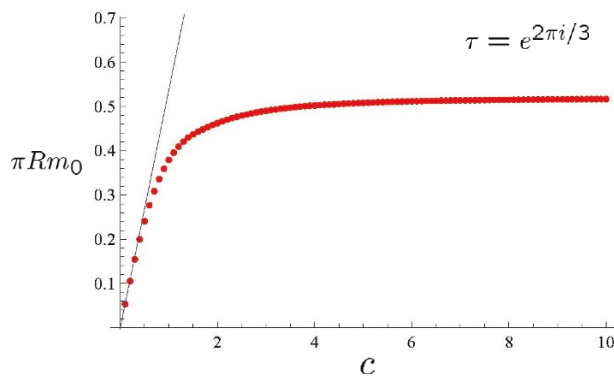


Figure 2. The lightest mass eigenvalue m_0 as a function of c in the case of $\tau = e^{2\pi i/3}$. The solid line represents (3.5).

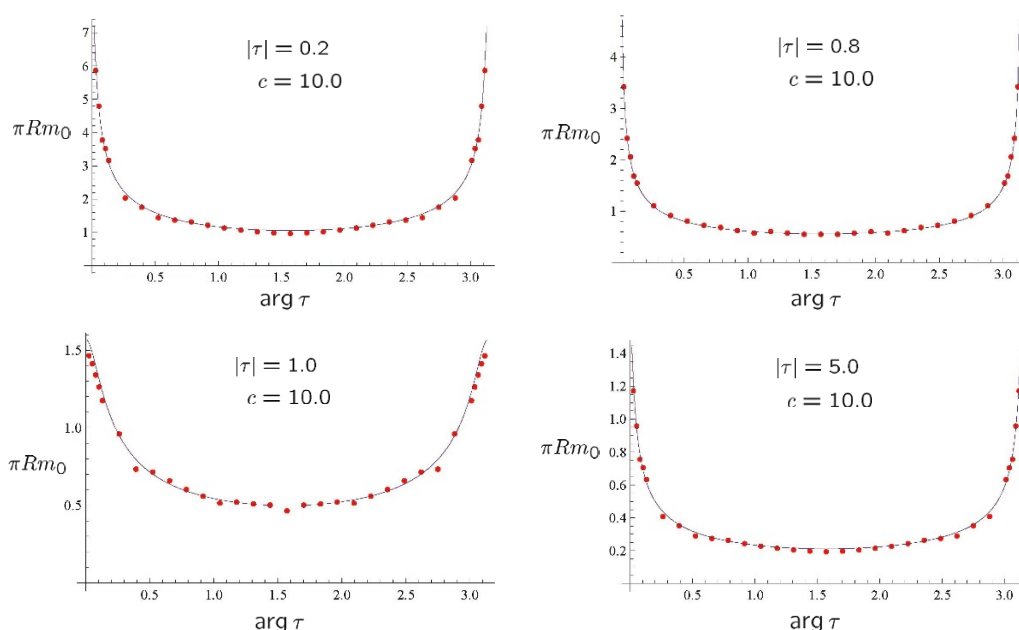


Figure 3. The lightest mass eigenvalue m_0 as a function of $\arg \tau$ for various values of $|\tau|$. The solid lines represent the approximate expression (4.12).

it is negligible. As the brane mass grows, such mixing effect becomes significant, and m_0 saturates and is almost independent of c when $c \gtrsim 5$. This situation is the same as the 5D case (see figure 5 in appendix A.1). In the following discussion, we take $c = 10.0$ as a representative of $c \gg 1$.

4 Approximate expression

Figure 3 shows the dependence of m_0 on $\arg \tau$ for various values of $|\tau|$. Here we will find an approximate expression of m_0 as a function of τ .

First, we should note that the mass eigenvalues m_a are functions of c and τ , and should satisfy

$$m_a \left(c; -\frac{1}{\tau} \right) = |\tau| m_a(c; \tau), \quad (4.1)$$

since the theory is defined on the torus. Besides, from (2.8) and (2.13), we also find that

$$m_a(c; -\bar{\tau}) = m_a(c; \tau). \quad (4.2)$$

As mentioned in the previous section, there are two limits in which the spacetime approaches to 5D, i.e., $\arg \tau \rightarrow 0, \pi$ (squashed torus) and $|\tau| \rightarrow 0, \infty$ (stretched torus). In these cases, the low-lying KK masses in the absence of the brane mass are approximately expressed as follows.

$|\tau| \gg 1$:

$$\tilde{m}_a = \tilde{m}_{n(a),0} \simeq \frac{|n(a)|}{R \operatorname{Im} \tau}, \quad (4.3)$$

where $a \lesssim 2|\tau|$, and $n(a) \equiv (-1)^a \operatorname{floor} \left(\frac{a+1}{2} \right)$.

$\theta \equiv \arg \tau \ll 0$:

$$m_{n,l} = \frac{1}{R} \left\{ \frac{(n + l|\tau| \cos \theta)^2}{|\tau|^2 \sin^2 \theta} + l^2 \right\}^{1/2} \simeq \frac{1}{R} \left\{ \frac{1}{\theta^2} \left(\frac{n}{|\tau|} + l \right)^2 + l^2 \right\}^{1/2}. \quad (4.4)$$

Especially when $|\tau|$ is a rational number, i.e., $|\tau| = p/q$ (p and q are relatively prime integers and $q > 0$), the light masses are approximated as

$$m_a = m_{n(a)p, -n(a)q} \simeq \frac{|n(a)q|}{R}, \quad (4.5)$$

where $a \lesssim \frac{2}{\theta} \min(1, |\tau^{-1} \pm 1|)$.

As for the cases of $|\tau| \ll 1$ and of $\pi - \arg \tau \ll 1$, approximate expressions of m_a are obtained from (4.3) and (4.5) by using (4.1) and (4.2), respectively. Then, we identify the effective radius of S^1 as

$$R_{\text{eff}} = \begin{cases} R \operatorname{Im} \tau & (|\tau| \gg 1) \\ R \operatorname{Im} \tau / |\tau| & (|\tau| \ll 1) \\ R/q & (\arg \tau \ll 1 \text{ or } \pi - \arg \tau \ll 1) \end{cases}. \quad (4.6)$$

Using this, the low-lying KK masses m_a can be expressed as (see appendix A.1)

$$m_a \simeq \frac{|n(a)|}{R_{\text{eff}}}, \quad (4.7)$$

or solutions of

$$m_a \simeq \frac{\hat{c}_{\text{eff}}^2}{2} \cot(\pi R_{\text{eff}} m_a), \quad (4.8)$$

where the “effective 5D brane mass” \hat{c}_{eff} is defined as

$$\hat{c}_{\text{eff}}^2 \equiv \frac{R_{\text{eff}} c^2}{2\pi R^2 \operatorname{Im} \tau}, \quad (4.9)$$

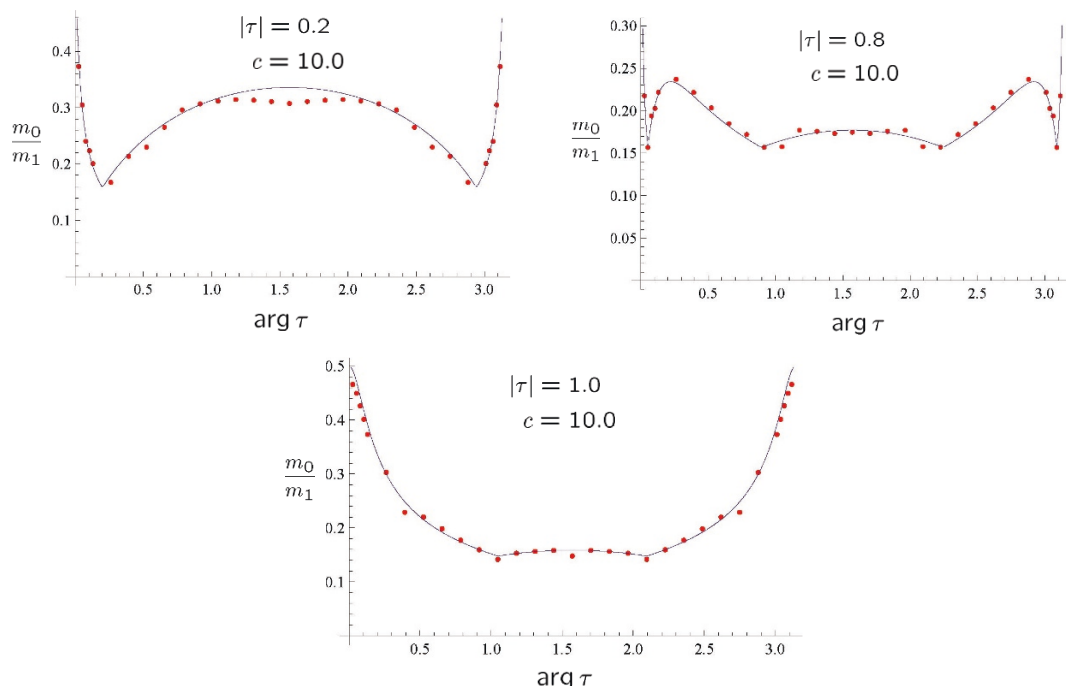


Figure 4. The ratio of the lightest mass eigenvalue m_0 to the compactification m_1 . The solid lines represents the ratio of (4.12) to (3.1).

which is identified from the condition that (3.5) is reproduced. When c is sufficiently large, the solutions of (4.8) are

$$m_a \simeq \frac{|n(a) + \frac{1}{2}|}{R_{\text{eff}}}. \quad (4.10)$$

Especially, the lightest mass eigenvalue is

$$m_0 \simeq \frac{1}{2R_{\text{eff}}} \simeq \frac{m_1}{2}. \quad (4.11)$$

Taking into account the properties (4.1), (4.2) and (4.11), we find an approximate expression of m_0 that fits figure 3 as

$$m_0^{(\text{ap})} = \frac{\sqrt{\sin \{ \arcsin(\tilde{\lambda}_1^2 \text{Im } \tau) \}}}{2\pi R \sqrt{\text{Im } \tau}}. \quad (4.12)$$

This is plotted as solid lines in figure 3.

Finally, we evaluate the ratio of m_0 to the compactification scale m_1 . As figure 4 shows, this ratio is much smaller than the value of the 5D case, $1/2$, except for the extreme cases in which the spacetime is 5D-like. Typically, m_0 is lighter than m_1 by one order of magnitude. Namely, the brane-localized mass cannot make the zero-modes as heavy as the compactification scale. This is an important fact in model building.

5 Summary and comments

5.1 Summary

We have evaluated the mass eigenvalues of a 6D theory compactified on a torus in the presence of the brane-localized mass term. Especially we focus on the lightest mode that becomes massless in the zero brane-mass limit.

From the numerical calculations, we confirmed that the lightest mass eigenvalue m_0 has non-negligible dependence on the cutoff scale Λ even when Λ is larger than the compactification scale by two orders of magnitude. This indicates that m_0 is sensitive to the internal structure of the brane when the brane has a finite size. This is consistent with the results in ref. [11].

We find an approximate expression of m_0 which is valid for a large brane mass. It clarifies the dependence on the size and the shape of the torus, and reduces to the known result in the 5D case when the torus is squashed or stretched.

In contrast to the 5D case, m_0 is much smaller than the compactification scale unless the torus is squashed or stretched. Their ratio is typically $\mathcal{O}(0.1)$. This is because the effects of the brane term are spread out over the codimension two compact space and diluted. Hence we should be careful in model building especially when we introduce the brane mass terms in order to decouple unwanted modes.

Although we have not discussed in this paper, the brane mass also deforms the profiles of the mode functions. They can be obtained by calculating the eigenvectors of M_{ab}^2 in (2.13). The main effect of the brane mass on the mode functions is to push them out from the position of the brane. Namely, it reduces their absolute values at the brane to zero.

5.2 On more general setups

We have discussed in a theory of a scalar field because it is the simplest case. However, the properties of the spectrum clarified in the text are also found in cases of fermion and vector fields, as shown in appendix B. So our result is valid in a wider class of 6D theory.

Besides, we have assumed that the bulk mass is zero and the brane squared mass is positive. In the presence of the bulk mass M_{bk} , the mass matrix (2.13) becomes

$$M_{ab}^2 = (M_{\text{bk}}^2 + \tilde{m}_a^2) \delta_{ab} + \frac{c^2}{4\pi^2 R^2 \text{Im } \tau}, \quad (5.1)$$

where $\tilde{m}_{n,l} = |n + l\tau| / (R \text{Im } \tau)$. Thus the bulk mass just raises the whole spectrum. However, if we allow a tachyonic brane mass, i.e., $c^2 < 0$, a light mode may appear below the compactification scale. If $|c|^2$ is large enough, m_0 becomes tachyonic and thus $\langle \phi \rangle = 0$ is no longer the vacuum. In such a case, ϕ has a nontrivial background that depend on the extra-dimensional coordinates z and \bar{z} , and we have to expand ϕ around it in order to obtain the mass matrix M_{ab}^2 . It is not an easy work to find such a nontrivial background. Here we do not discuss this issue further, but give a comment on it. Note that the smallest diagonal element $M_{00}^2 = M_{\text{bk}}^2 + c^2 / (4\pi^2 R^2 \text{Im } \tau)$ provides the upper bound on m_0 . Thus, $2\pi R \sqrt{\text{Im } \tau} M_{\text{bk}} > |c|$ must be satisfied in order to avoid the vacuum instability for $\langle \phi \rangle = 0$.

In other words, there is a value of c that leads to a tachyonic mass eigenvalue no matter how large M_{bk} is. This indicates that the effect of the brane mass on the spectrum does not saturate, which is in contrast to the non-tachyonic brane mass. The mode function is attracted toward the brane by the tachyonic brane mass.

We considered the scalar field with the periodic boundary condition. Twisted boundary conditions are also allowed, but they just raise the mass spectrum. This can be understood from the fact that imposing the twisted boundary conditions is equivalent to introducing a non-vanishing background gauge field coupled to the scalar field with the periodic boundary conditions. Such a background gauge field play the same role as the bulk scalar mass M_{bk} mentioned above.

We have also assumed that the spacetime is flat, no background magnetic fluxes exist,⁴ and there is only one brane, for simplicity. It is an interesting and useful extension to relax these assumptions. This will be discussed in separate papers.

Acknowledgments

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A 5D case

Here we summarize the effects of the brane-localized mass in a 5D complex scalar theory. The Lagrangian is

$$\mathcal{L} = -\partial^{\hat{M}}\phi^*\partial_{\hat{M}}\phi - \hat{c}^2|\phi|^2\delta(x^4) + \dots, \quad (\text{A.1})$$

where $\hat{M} = 0, 1, 2, 3, 4$, and the ellipsis denotes interaction terms. The brane mass parameter \hat{c} is a real dimension 1/2 constant. The extra dimension is compactified on S^1 whose radius is R .

A.1 Analytic expressions

The KK expansion of ϕ is

$$\phi(x^\mu, x^4) = \sum_{n=-\infty}^{\infty} f_n(x^4)\phi_n(x^\mu). \quad (\text{A.2})$$

The mode function $f_n(x^4)$ satisfies the mode equation, which is read off from (A.1) as

$$\{\partial_4^2 - \hat{c}^2\delta(x^4)\} f_n(x^4) = -m_n^2 f_n(x^4). \quad (\text{A.3})$$

By integrating this over an infinitesimal interval $[-\epsilon, \epsilon]$, we obtain

$$[\partial_4 f_n]_{-\epsilon}^{\epsilon} - \hat{c}^2 f_n(0) = 0. \quad (\text{A.4})$$

⁴The introduction of the background fluxes leads to the multiplication of the modes at each KK level. Thus the size of the mass matrix (2.13) becomes larger, and it will take much more time to calculate the mass eigenvalues. So we need to develop more efficient way to discuss in such a case.

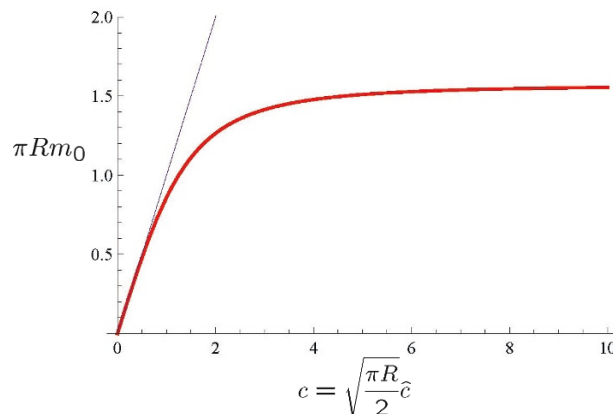


Figure 5. The lightest mass eigenvalue m_0 as a function of \hat{c} .

Thus the brane mass changes the boundary condition of the bulk field.

In the bulk region $[\epsilon, 2\pi R - \epsilon]$, (A.3) is solved as

$$f_n(x^4) = C_{+n} e^{im_n x^4} + C_{-n} e^{-im_n x^4}. \quad (\text{A.5})$$

where $C_{\pm n}$ are complex constants. From the periodic condition $f_n(x^4 + 2\pi R) = f_n(x^4)$ and (A.4), we obtain

$$\begin{aligned} C_{+n} + C_{-n} &= C_{+n} e^{2\pi i m_n R} + C_{-n} e^{-2\pi i m_n R}, \\ im_n (C_{+n} - C_{-n}) - im_n (C_{+n} e^{2\pi i m_n R} - C_{-n} e^{-2\pi i m_n R}) &= \hat{c}^2 (C_{+n} + C_{-n}). \end{aligned} \quad (\text{A.6})$$

The solutions of these equations are

$$C_{-n} = -C_{+n}, \quad m_n = \frac{|n|}{R}, \quad (n \neq 0) \quad (\text{A.7})$$

or

$$C_{-n} = e^{2\pi i m_n R} C_{+n}, \quad m_n = \frac{\hat{c}^2}{2} \cot(\pi R m_n). \quad (\text{A.8})$$

When $\hat{c} = 0$, the spectrum determined by the second equation of (A.8) coincides with that of (A.7). Note that the mode functions corresponding to (A.7) are odd functions. Thus, they do not feel the brane-localized mass because they vanish at $x^4 = 0$.

Figure 5 shows the lightest mass m_0 as a function of the brane mass \hat{c} . For small values of the brane mass, m_0 is proportional to \hat{c} . This is because the brane mass can be treated as a perturbation in this region, and the mixing among the KK modes induced by it is negligible. As the brane mass grows, such mixing effect becomes significant, and m_0 saturates and is almost independent of \hat{c} when $c \equiv \sqrt{\pi R/2} \hat{c} \gtrsim 5$. In the limit of $\hat{c} \rightarrow \infty$, the spectrum determined by (A.8) is

$$m_n = \frac{|n + \frac{1}{2}|}{R}. \quad (\text{A.9})$$

Since the second smallest solution of the second equation in (A.8) are greater than $1/R$, the first KK excited mass is $m_1 = 1/R$, which is independent of \hat{c} and taken as the compactification scale.

A.2 Numerical evaluation

In order to see the cutoff dependence of the spectrum and compare it with that in the 6D case, we follow the same procedure as in section 3.

We relabel the KK modes by using the KK label $a = 0, 1, 2, \dots$, which is defined as

$$a = \begin{cases} 2n & (n \geq 0) \\ 2|n| - 1 & (n < 0) \end{cases}, \quad (\text{A.10})$$

and expand ϕ as

$$\phi(x^\mu, x^4) = \sum_{a=0}^{\infty} \frac{e^{in(a)\pi x^4/R}}{\sqrt{2\pi R}} \phi_a(x^\mu), \quad (\text{A.11})$$

where $n(a) \equiv (-1)^a \text{floor}(\frac{a+1}{2})$. This is the KK expansion in the absence of the brane mass. Then, we can rewrite the 5D Lagrangian (A.1) in terms of ϕ_a as

$$\mathcal{L}^{(4D)} = - \sum_a \partial^\mu \phi^* \partial_\mu \phi - \sum_{a,b} \hat{M}_{ab}^2 \phi_a^* \phi_b + \dots, \quad (\text{A.12})$$

where

$$\hat{M}_{ab}^2 \equiv \left(\frac{n(a)}{R} \right)^2 \delta_{ab} + \frac{\hat{c}^2}{2\pi R}. \quad (\text{A.13})$$

We consider only the modes whose masses are below the cutoff scale Λ .

$$0 < \frac{|n(N_\Lambda)|}{R} < \Lambda \leq \frac{|n(N_\Lambda + 1)|}{R}. \quad (\text{A.14})$$

Then the KK mass eigenvalues $\{m_0^2, m_1^2, \dots, m_{N_\Lambda}^2\}$ are calculated as eigenvalues of the finite matrix \hat{M}_{ab}^2 ($a, b = 0, 1, \dots, N_\Lambda$).

Figure 6 shows the lightest mass eigenvalue m_0 as a function of Λ in the unit of $m_1 = 1/R$ when $c = 100$. The solid line represents

$$\frac{m_0(\Lambda)}{m_1} = \left(3.01 + 9.87 \frac{\Lambda}{m_1} \right)^{-1} + 0.500. \quad (\text{A.15})$$

We can see from figure 6 that m_0 rapidly approaches to $m_1/2 = 1/(2R)$, which is consistent with (A.9). The cutoff dependence is negligible when $\Lambda \gtrsim 10m_1$.

B Cases of spinor and vector fields

B.1 Brane mass for spinor fields

We consider a theory which has a 6D Weyl spinor field Ψ_+ whose 6D chirality is $+$. We can introduce the following brane mass term with the 4D spinor field localized on the brane.

$$\mathcal{L} = i\bar{\Psi}_+ \Gamma^M \partial_M \Psi_+ + \{ -i\chi \sigma^\mu \partial_\mu \bar{\chi} + c(\psi\chi + \bar{\psi}\bar{\chi}) \} \delta(x^4) \delta(x^5), \quad (\text{B.1})$$

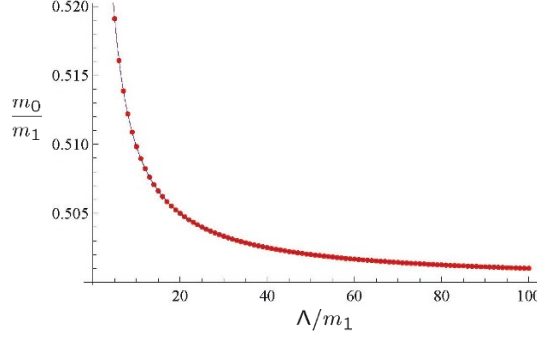


Figure 6. The lightest mass eigenvalue m_0 as a function of Λ in the unit of the compactification scale m_1 . The brane mass is chosen as $c = 100$.

where χ is a 4D left-handed Weyl spinor, and the 2-component spinor ψ is the 4D right-handed component of the 4-component spinor $\hat{\Psi}$, which is defined as

$$\Psi_+ \equiv \begin{pmatrix} \hat{\Psi} \\ 0 \end{pmatrix}. \quad (\text{B.2})$$

The 6D gamma matrices Γ^M are defined as

$$\Gamma^\mu = \begin{pmatrix} \gamma^\mu & \\ & \gamma^\mu \end{pmatrix}, \quad \Gamma^4 = \begin{pmatrix} & i\gamma_5 \\ i\gamma_5 & \end{pmatrix}, \quad \Gamma^5 = \begin{pmatrix} & \mathbf{1}_4 \\ -\mathbf{1}_4 & \end{pmatrix}. \quad (\text{B.3})$$

The brane mass parameter c is dimensionless, and assumed to be real.

In the 2-component notation, (B.1) is rewritten as

$$\begin{aligned} \mathcal{L} = & -i\lambda\sigma^\mu\partial_\mu\bar{\lambda} - i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi + \frac{1}{\pi R}\lambda\partial_{\bar{z}}\psi - \frac{1}{\pi R}\bar{\psi}\partial_z\bar{\lambda} \\ & + \frac{1}{2\pi^2 R^2} \left\{ -i\chi\sigma^\mu\partial_\mu\bar{\chi} + c(\psi\chi + \bar{\psi}\bar{\chi}) \right\} \delta^{(2)}(z), \end{aligned} \quad (\text{B.4})$$

where λ is the 4D left-handed component of $\hat{\Psi}$. Thus, the equations of motion are

$$\begin{aligned} -i\sigma^\mu\partial_\mu\bar{\lambda} + \frac{1}{\pi R}\partial_{\bar{z}}\psi &= 0, \\ -i\bar{\sigma}^\mu\partial_\mu\psi - \frac{1}{\pi R}\partial_z\bar{\lambda} + \frac{c}{2\pi^2 R^2}\chi\delta^{(2)}(z) &= 0, \\ -i\sigma^\mu\partial_\mu\bar{\chi} + c\psi|_{z=0} &= 0. \end{aligned} \quad (\text{B.5})$$

From these, we obtain

$$\left\{ \square_4 + \frac{1}{\pi^2 R^2} \partial_z \partial_{\bar{z}} - \frac{c^2}{2\pi^2 R^2} \delta^{(2)}(z) \right\} \psi = 0. \quad (\text{B.6})$$

This has the same form as the equation of motion for ϕ derived from (2.3). Hence the spectrum in this system is the same as that of the scalar field discussed in the text [11].

In the case that Ψ_+ does not have any charges, the following Majorana mass term is allowed on the brane.

$$\mathcal{L} = i\bar{\Psi}_+\Gamma^M\partial_M\Psi_+ + h(\psi^2 + \bar{\psi}^2)\delta(x^4)\delta(x^5), \quad (\text{B.7})$$

where the brane mass parameter h has the mass dimension -1 . In contrast to the above Dirac mass term, the equation of motion in this case does not have the form of (B.6). Thus the spectrum in this case has to be discussed separately.

B.2 Brane mass for vector field

Here we consider a theory of a vector field A^M . We can introduce the following brane mass term.

$$\mathcal{L} = -\frac{1}{4}F^{MN}F_{MN} - \frac{1}{2}(\partial^\mu S + cA^\mu)(\partial_\mu S + cA_\mu)\delta(x^4)\delta(x^5), \quad (\text{B.8})$$

where the real scalar S is the Stueckelberg's scalar field, which is localized on the brane. The brane mass parameter c is real and dimensionless. The above Lagrangian is invariant under the gauge transformation:

$$\begin{aligned} A_M(x, z) &\rightarrow A_M(x, z) + \partial_M\Lambda(x, z), \\ S(x) &\rightarrow S(x) - c\Lambda(x, 0), \end{aligned} \quad (\text{B.9})$$

where $\Lambda(x, z)$ is the gauge transformation parameter.

In order to fix the gauge, we add the following gauge-fixing term.

$$\mathcal{L}_{\text{gf}} = -\frac{1}{2\xi} \left\{ \partial^\mu A_\mu + \frac{\xi}{2\pi^2 R^2} (\partial_z A_{\bar{z}} + \partial_{\bar{z}} A_z) \right\}^2, \quad (\text{B.10})$$

where ξ is the gauge parameter.

Performing the partial integral, the total Lagrangian becomes

$$\begin{aligned} \mathcal{L} + \mathcal{L}_{\text{gf}} = & -\frac{1}{2} \left\{ \partial^\mu A^\nu \partial_\mu A_\nu - \left(1 - \frac{1}{\xi}\right) \partial^\mu A^\nu \partial_\nu A_\mu + \frac{1}{\pi^2 R^2} \partial_z A^\mu \partial_{\bar{z}} A_\mu \right\} \\ & - \frac{1}{2\pi^2 R^2} \left\{ \partial^\mu A_z \partial_\mu A_{\bar{z}} - \frac{1-\xi}{4\pi^2 R^2} ((\partial_z A_{\bar{z}})^2 + (\partial_{\bar{z}} A_z)^2) + \frac{1+\xi}{2\pi^2 R^2} |\partial_z A_{\bar{z}}|^2 \right\} \\ & - \frac{1}{4\pi^2 R^2} (c^2 A^\mu A_\mu - 2cS\partial^\mu A_\mu + \partial^\mu S\partial_\mu S) \delta^{(2)}(z). \end{aligned} \quad (\text{B.11})$$

We expand the 6D gauge field into the 4D KK modes as

$$\begin{aligned} A_\mu(x, z) &= \sum_a u_a(z) A_\mu^{(a)}(x) + \sum_a w_a(z) \partial_\mu A_S^{(a)}(x), \\ A_z(x, z) &= \sqrt{2\pi R} \sum_a v_a(z) \varphi^{(a)}(x), \end{aligned} \quad (\text{B.12})$$

where $A_\mu^{(a)}(x)$ satisfies $\partial^\mu A_\mu^{(a)}(x) = 0$. The mode functions $u_a(z)$ and $w_a(z)$ are real, but $v_a(z)$ is complex. They are normalized as⁵

$$\int_{T^2} d^2z u_a^2(z) = \int_{T^2} d^2z w_a^2(z) = \int_{T^2} d^2z |v_a(z)|^2 = \frac{1}{2\pi^2 R^2}. \quad (\text{B.13})$$

⁵Note that $dx^4 dx^5 = 2\pi^2 R^2 d^2z$.

Here we choose the mode functions as

$$\begin{aligned}
 u_0(z) &= w_0(z) = \frac{1}{2\pi R\sqrt{\text{Im } \tau}}, & u_{2\hat{a}}(z) &= w_{2\hat{a}}(z) = \frac{\cos\{2\text{Im}(\lambda_{\hat{a}}z)\}}{\pi R\sqrt{2\text{Im } \tau}}, \\
 u_{2\hat{a}-1}(z) &= w_{2\hat{a}-1}(z) = \frac{\sin\{2\text{Im}(\lambda_{\hat{a}}z)\}}{\pi R\sqrt{2\text{Im } \tau}}, \\
 v_{n,l}(z) &= \frac{e^{\lambda_{n,l}z - \lambda_{n,l}^*\bar{z}}}{2\pi R\sqrt{\text{Im } \tau}},
 \end{aligned} \tag{B.14}$$

where $a = (n, l)$ (see section 2), $\hat{a} = 1, 2, \dots$, and

$$\lambda_{n,l} \equiv \frac{\pi(n + l\bar{\tau})}{\text{Im } \tau}. \tag{B.15}$$

Then, in terms of the KK modes, (B.11) becomes

$$\begin{aligned}
 \mathcal{L}^{(4D)} &\equiv \int_{T^2} dx^4 dx^5 (\mathcal{L} + \mathcal{L}_{\text{gf}}) = 2\pi^2 R^2 \int d^2 z (\mathcal{L} + \mathcal{L}_{\text{gf}}) \\
 &= \frac{1}{2} \sum_a A^{(a)\mu} \left(\square_4 - \frac{|\lambda_a|^2}{\pi^2 R^2} \right) A_\mu^{(a)} + \frac{1}{2} \sum_a \partial^\mu A_S^{(a)} \left(\frac{\square_4}{\xi} - \frac{|\lambda_a|^2}{\pi^2 R^2} \right) \partial_\mu A_S^{(a)} \\
 &\quad - \frac{c^2}{2} \sum_{a,b} u_a(0) u_b(0) \left\{ A^{(a)\mu} A_\mu^{(b)} + \partial^\mu A_S^{(a)} \partial_\mu A_S^{(b)} \right\} \\
 &\quad + c \sum_a u_a(0) S \square_4 A_S^{(a)} - \frac{1}{2} \partial^\mu S \partial_\mu S \\
 &\quad - \sum_a \left\{ \partial^\mu \varphi^{(a)*} \partial_\mu \varphi^{(a)} + \frac{(1+\xi)|\lambda_a|^2}{2\pi^2 R^2} |\varphi^{(a)}|^2 \right\} \\
 &\quad + \sum_{n,l} \frac{1-\xi}{2\pi^2 R^2} \text{Re} \left\{ \lambda_{n,l}^{*2} \varphi^{(n,l)} \varphi^{(-n,-l)} \right\}.
 \end{aligned} \tag{B.16}$$

We have used that

$$\int d^2 z v_{n,l}(z) v_{n',l'}(z) = \frac{\delta_{n,-n'} \delta_{l,-l'}}{2\pi^2 R^2}. \tag{B.17}$$

Furthermore, we decompose $\varphi^{(n,l)}(x)$ as

$$\varphi^{(n,l)} = \frac{e^{i\theta_{n,l}}}{\sqrt{2}} \left(\varphi_R^{(n,l)} + i\varphi_I^{(n,l)} \right), \tag{B.18}$$

where $\theta_{n,l} \equiv \arg(\lambda_{n,l})$. Then, the last two lines in (B.16) is rewritten as

$$\begin{aligned}
 \mathcal{L}_\varphi^{(4D)} &= -\frac{1}{2} \sum_a \left\{ \partial^\mu \varphi_R^{(a)} \partial_\mu \varphi_R^{(a)} + \partial^\mu \varphi_I^{(a)} \partial_\mu \varphi_I^{(a)} + \frac{(1+\xi)|\lambda_a|^2}{2\pi^2 R^2} \left(\varphi_R^{(a)2} + \varphi_I^{(a)2} \right) \right\} \\
 &\quad - \frac{1-\xi}{4\pi^2 R^2} \sum_{n,l} |\lambda_{n,l}|^2 \left(\varphi_R^{(n,l)} \varphi_R^{(-n,-l)} - \varphi_I^{(n,l)} \varphi_I^{(-n,-l)} \right).
 \end{aligned} \tag{B.19}$$

Thus, the equations of motion are

$$\begin{aligned}
 \sum_b \left(\square_4 \delta^{ab} - M_{Aab}^2 \right) A_\mu^{(b)} &= 0, \\
 \square_4 \left\{ \sum_b \left(\square_4 \delta^{ab} - \xi M_{Aab}^2 \right) A_S^{(b)} - \xi c u_a(0) S \right\} &= 0, \\
 \square_4 S + c \sum_a u_a(0) \square_4 A_S^{(a)} &= 0, \\
 \left\{ \square_4 - \frac{(1+\xi) |\lambda_{n,l}|^2}{2\pi^2 R^2} \right\} \varphi_R^{(n,l)} - \frac{(1-\xi) |\lambda_{n,l}|^2}{2\pi^2 R^2} \varphi_R^{(-n,-l)} &= 0, \\
 \left\{ \square_4 - \frac{(1+\xi) |\lambda_{n,l}|^2}{2\pi^2 R^2} \right\} \varphi_I^{(n,l)} + \frac{(1-\xi) |\lambda_{n,l}|^2}{2\pi^2 R^2} \varphi_I^{(-n,-l)} &= 0,
 \end{aligned} \tag{B.20}$$

where

$$\begin{aligned}
 M_{Aab}^2 &\equiv \frac{|\lambda_a|^2}{\pi^2 R^2} \delta_{ab} + c^2 u_a(0) u_b(0) \\
 &= \begin{cases} \frac{1}{\pi^2 R^2} \left(|\lambda_a|^2 \delta_{ab} + \frac{c^2}{4 \text{Im } \tau} \right) & (a = b = 0) \\ \frac{1}{\pi^2 R^2} \left(|\lambda_a|^2 \delta_{ab} + \frac{c^2}{2 \text{Im } \tau} \right) & (a \text{ and } b : \text{even but } (a, b) \neq (0, 0)) \\ \frac{|\lambda_a|^2}{\pi^2 R^2} \delta_{ab} & (a \text{ or } b : \text{odd}) \end{cases} \tag{B.21}
 \end{aligned}$$

From the second and the third equations of (B.20), we can eliminate S and obtain

$$\square_4 \left(\square_4 - \frac{\xi |\lambda_a|^2}{\pi^2 R^2} \right) A_S^{(a)} = 0. \tag{B.22}$$

and from the last two equations of (B.20), we obtain

$$\begin{aligned}
 \left(\square_4 - \frac{|\lambda_a|^2}{\pi^2 R^2} \right) \varphi_{R+}^{(a)} &= 0, & \left(\square_4 - \frac{\xi |\lambda_a|^2}{\pi^2 R^2} \right) \varphi_{R-}^{(a)} &= 0, \\
 \left(\square_4 - \frac{\xi |\lambda_a|^2}{\pi^2 R^2} \right) \varphi_{I+}^{(a)} &= 0, & \left(\square_4 - \frac{|\lambda_a|^2}{\pi^2 R^2} \right) \varphi_{I-}^{(a)} &= 0,
 \end{aligned} \tag{B.23}$$

where

$$\begin{aligned}
 \varphi_{R\pm}^{(n,l)} &\equiv \frac{1}{\sqrt{2}} \left(\varphi_R^{(n,l)} \pm \varphi_R^{(-n,-l)} \right), \\
 \varphi_{I\pm}^{(n,l)} &\equiv \frac{1}{\sqrt{2}} \left(\varphi_I^{(n,l)} \pm \varphi_I^{(-n,-l)} \right).
 \end{aligned} \tag{B.24}$$

Therefore, we can see that the spectrum for $A_\mu^{(2\hat{a})}(x)$ ($\hat{a} = 0, 1, \dots$) has similar properties to that of the scalar field discussed in the text. In contrast, the spectra for $A_\mu^{(2\hat{a}-1)}(x)$

($\hat{a} = 1, 2, \dots$), $\varphi_{R+}^{(a)}(x)$ and $\varphi_{I-}^{(a)}(x)$ do not receive an effect of the brane mass, and are given by

$$m_a = \frac{|\lambda_a|}{\pi R} = \frac{\pi |n + l\tau|}{\pi R \text{Im } \tau}. \quad (\text{B.25})$$

This is the result from the fact that the mode functions of $A_\mu^{(2\hat{a}-1)}(x)$ vanish at the brane, and $A_z(x, z)$ does not have a brane mass in the first place. The remaining modes are unphysical, and their spectra depend on the gauge parameter ξ , i.e.,

$$m_a = \frac{\sqrt{\xi} \pi |n + l\tau|}{\pi R \text{Im } \tau}. \quad (\text{B.26})$$

Especially, $\varphi_{R-}^{(a)}(x)$ and $\varphi_{I+}^{(a)}(x)$ are the would-be NG modes, which are absorbed into the longitudinal modes of the massive KK modes for A_μ .

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